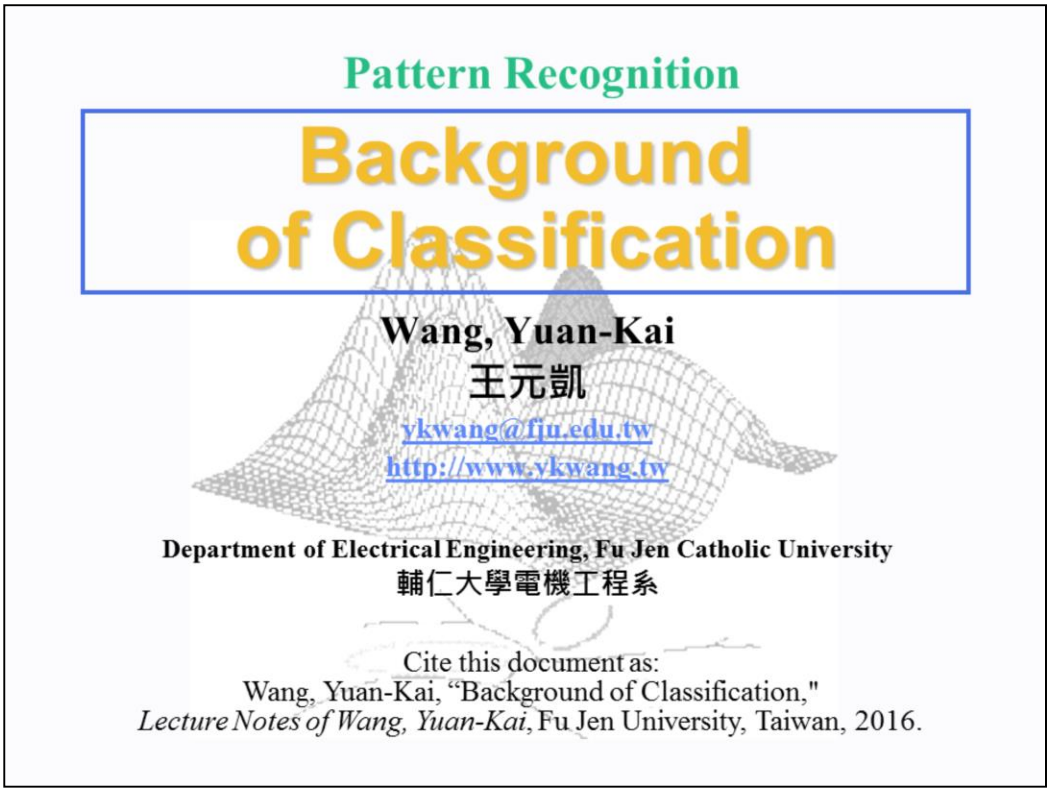



- This is the second lecture note of the course PATTERN RECOGNITION in English in 104-2 semester, EE, FJU.
- In this lecture note, I will introduce mathematical basics classification.
- Web site of this course: http://pattern-recognition.weebly.com.

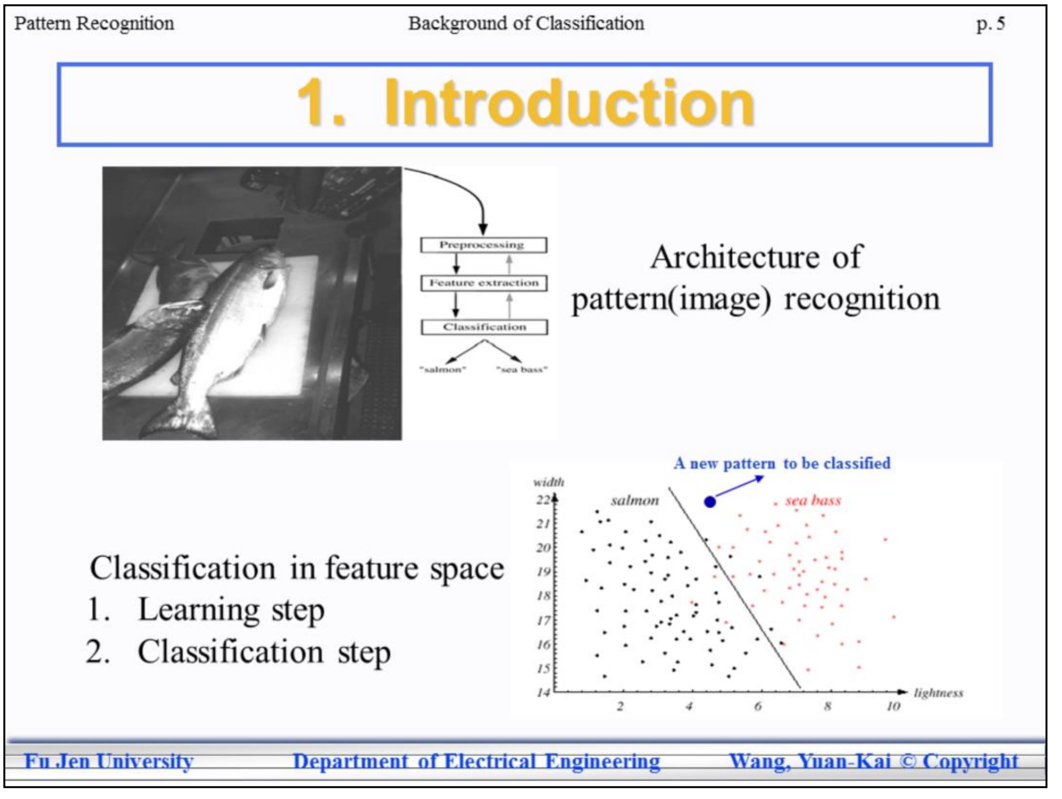
# Goal of This Unit

- Get familiar with basic concepts with respect to "feature space"
- Know "separable patterns"
- Understand the roles of "linear" and "nonlinear" functions to classify patterns
- Get acquainted with "machine learning" and "neural network"

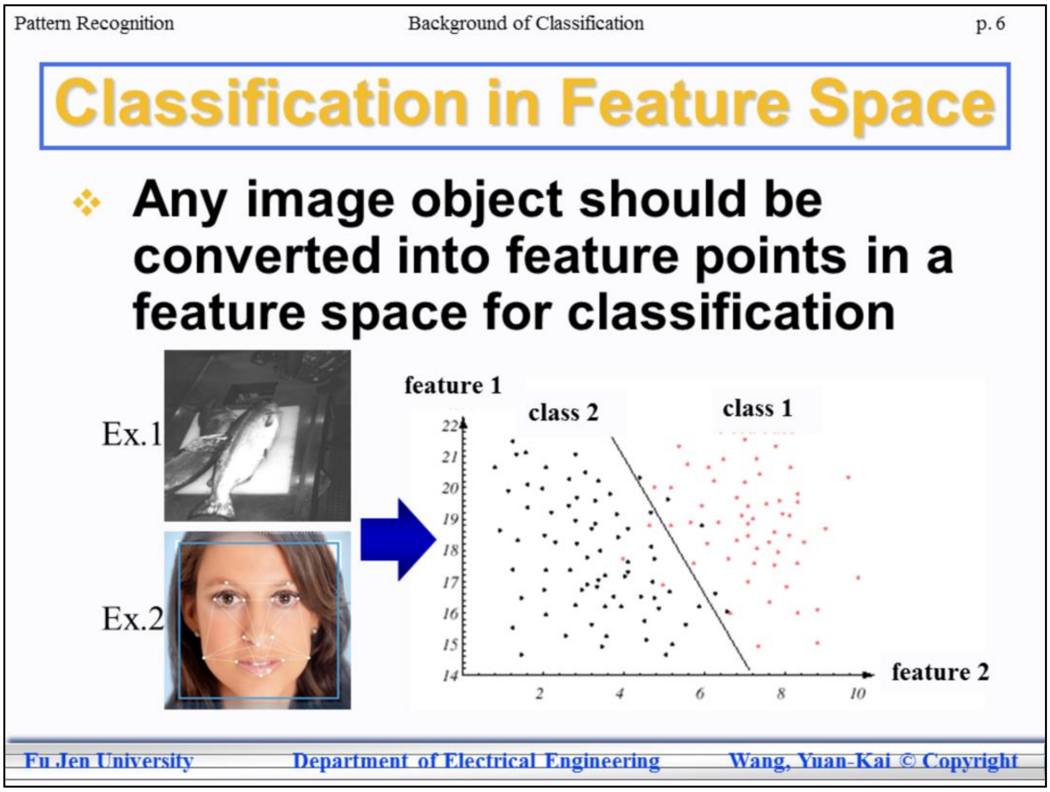
# References

- There is no reference for this unit
- But the following online book is helpful
  - Celebi Tutorial: Neural Networks and Pattern Recognition Using MATLAB (https://www.byclb.com/TR/Tutorials/neural_networks/)
    - Chapter 1 Pattern Classification
    - Chapter 2 Matrix Theory and Applications with Matlab
    - Chapter 8 Classical Models of Neural Network
    - Chapter 9 Linear Discriminant Functions

# Contents



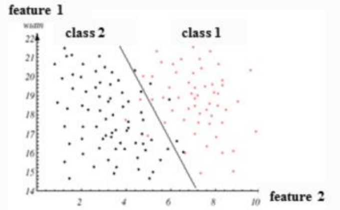

- Last unit we know the processing pipeline of an image recognition system
  - Preprocessing step: process the image the denoise and enhance objects
  - Feature extraction step: extract object's features
  - Classification step: classify the objects into a class
- Classification in feature space includes two steps
  - Learning step: given a lot of patterns in feature space (black and red dots), find the line that separates the patterns.
  - Classification step: given a new pattern with unknown class (one large blue dot), find the class of the given pattern by the line.
- We will explore more in this unit for the "classification" step.




- We classify image objects, such as fishes and faces, by their distribution in an algebraic space called feature space.
  - Although practically fish classification and face recognition are different problems
  - Theoretically we think they are the same problem: classify objects in their feature spaces
- In this feature space with a two-class problem
  - Each axis represents a feature
  - Each dot represents an image object
  - A class of image objects is assumed to be clustered in a region
  - A class consists a set of object patterns/object points
  - A line/curve is called a classifier (or a classification method) if it separates the space into two regions and separates the image objects into two sets.

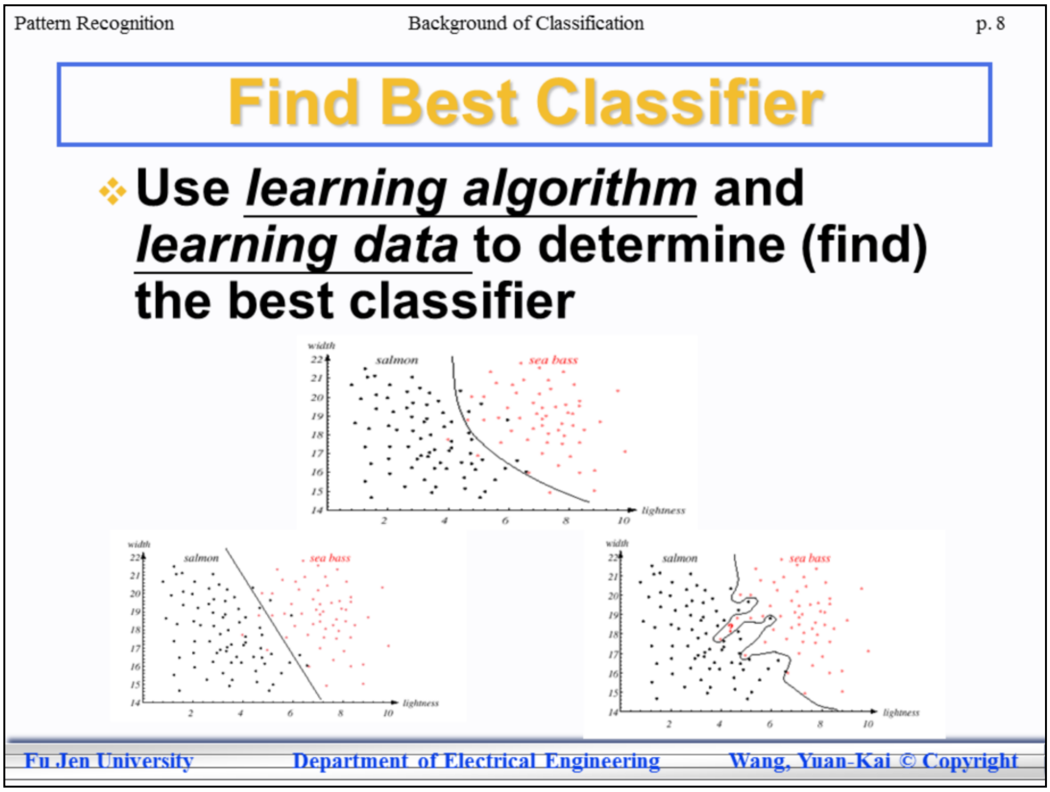
# Concepts and Terminologies

- Feature space
- Patterns in feature space
  - Linearly separable pattern
  - Nonlinearly separable pattern
- Discriminant and classifier
  - Minimum distance classifier






- In this unit, we will explore concepts and terminologies of classification in feature space
- First I will explain "feature space" in Section 2
- Section 3 gives some examples of patterns in feature space
  - Linear patterns
  - Nonlinear patters
- Section 4 explains classifier as discriminants.
  - The straight line separating the two classes is called : discriminant, or classifier
  - The concept of minimum distance classifier is also introduced.
- Section 5 explains machine learning for pattern recognition.



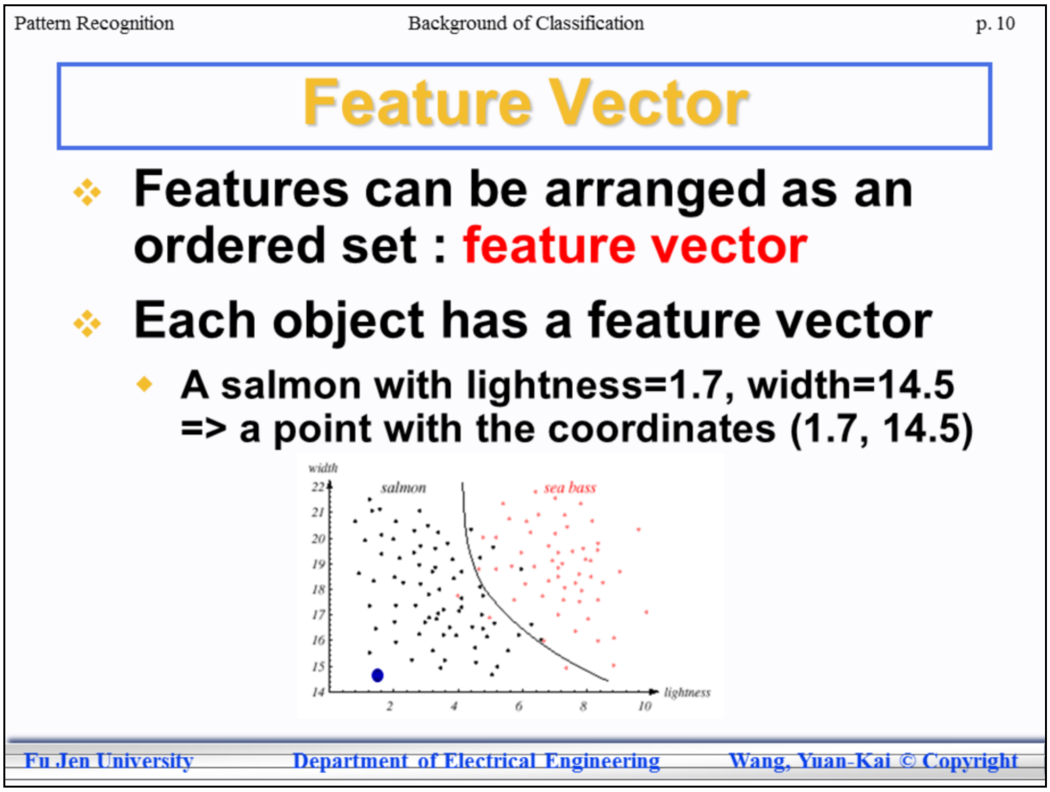

- For a classification problem, there are infinite classifiers
    - Linear classifier and nonlinear classifier
    - But we may have only one "best" classifier
- How can a computer program "automatically" find the best classifier? We need two things
    - Learning (training) algorithm
    - Learning (training) data
- What is a learning algorithm
    - Learning algorithm uses learning data to find the best classifier.
- But how?
    - Remember that each classifier has an error rate. And the best classifier has the minimum error rate.
    - We have infinite number of classifiers: infinite straight lines and infinite curves. Each classifier has an error rate.
    - We can find the best classifier only if we calculate all of the error rates of classifiers and find the minimum of these error rate values.
    - But it is a mission impossible.
    - Therefore a lot of complex algorithm are developed to conquer this difficulty.
- This unit will focus on "classification in feature space" only.
- Learning algorithms will not be explained in this unit, but will be explained in next unit.

# 2. Feature Space

- The objects that we are trying to classify are represented by **features**
- Features span a multidimensional space called **feature space**
  - **An axe** represents **a feature**
  - **A point** represents **an object**, whose coordinates are the values of the features



- This section will explain what is feature space.




- A feature vector $p_1$ is a point in a $d$-dimension pattern space.
- Each element of the vector is a feature, and each one corresponds to one dimension (axis) in the space.
- In the fish classification example, we have only two features
  - The feature space is a 2-dimensional space: d=2.
  - Each fish is denoted as $p_1$.

# Math Definition

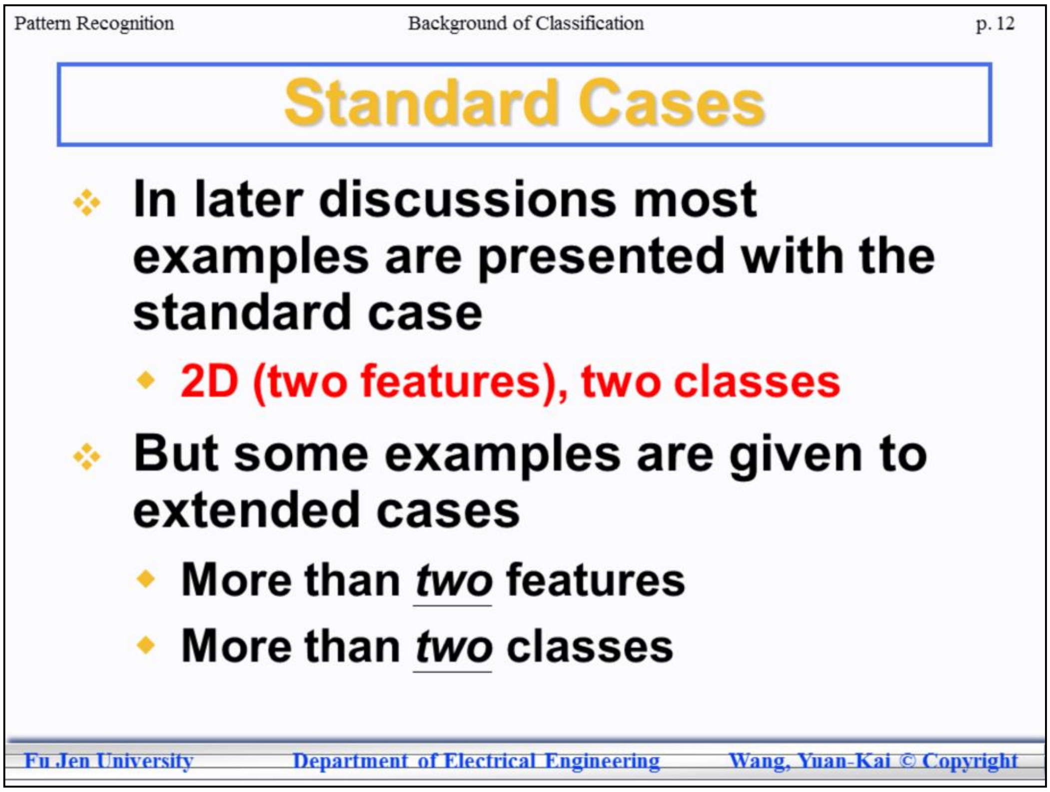
- Suppose there are $n$ features, $x_i$, $i=1,2\ldots,n$, to describe the object
  - $n$-dimensional feature space
- A feature vector **X** is defined as

  $$\mathbf{X} = [x_1, x_2, \ldots x_n]^T$$

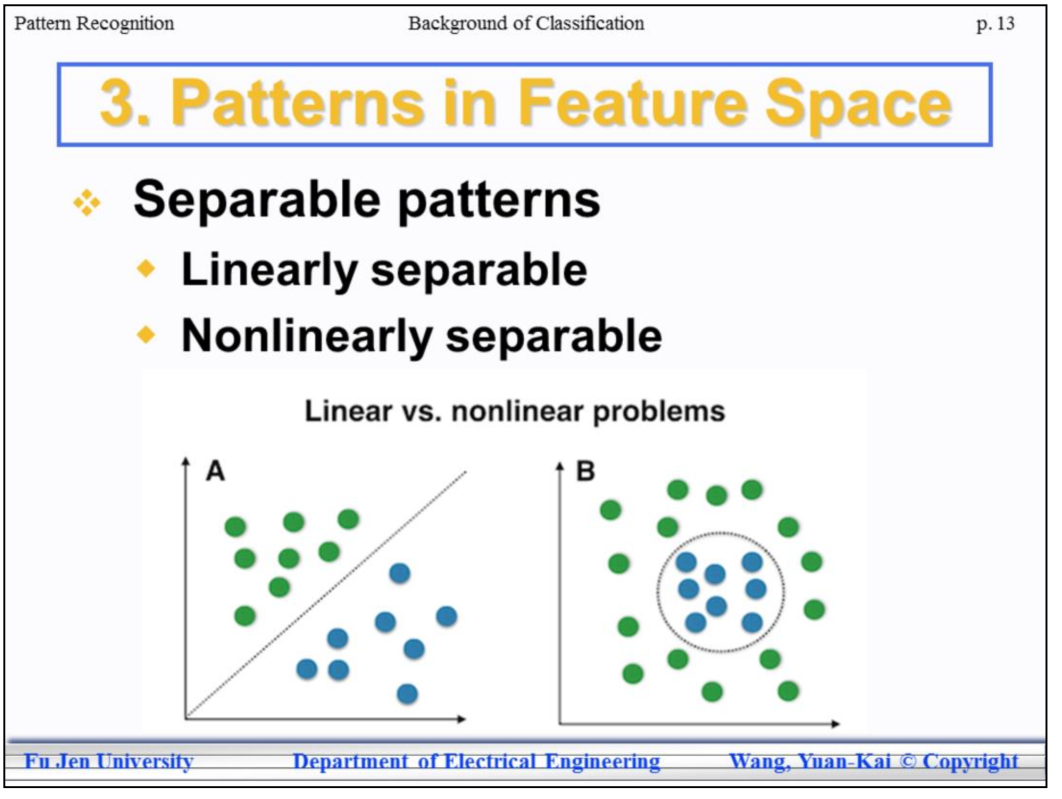
  - Each of the feature vectors represents a single object (pattern)
- Features and feature vectors can also be treated as random variables and random vectors



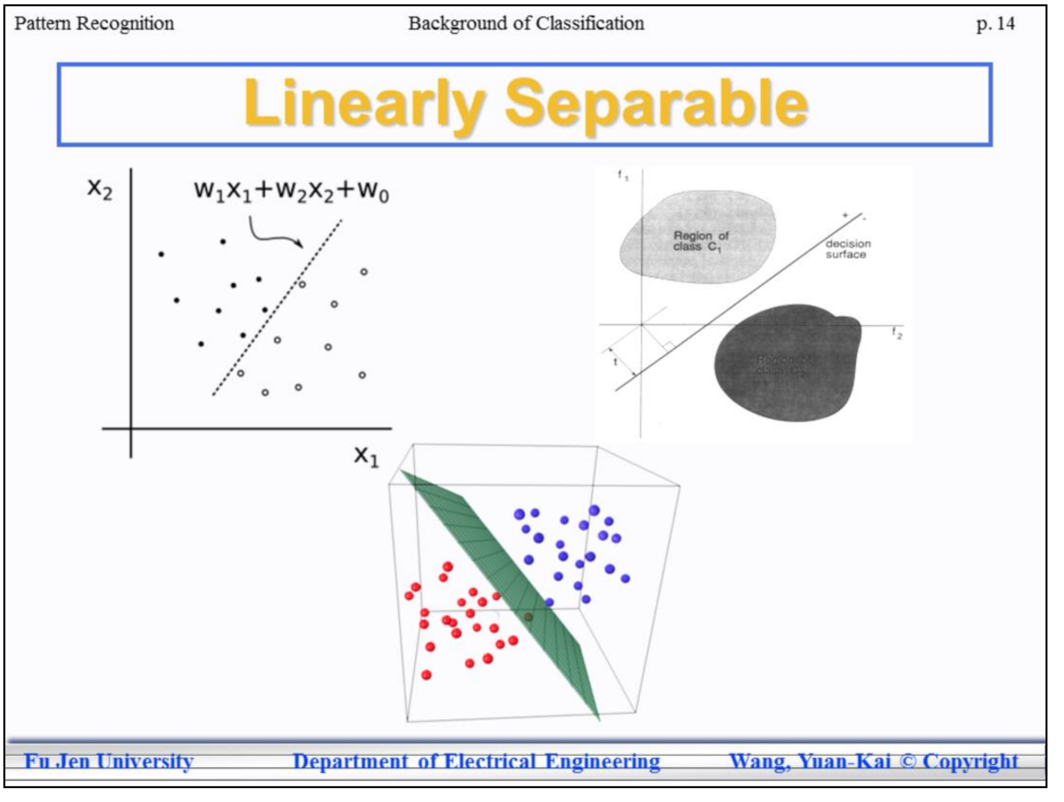
- Mathematically we can easily extend a pattern classification problem into n-dimensional space
- We then have two mathematical tools to help us solve classification problems
  - Linear algebra: if we consider the feature space as an algebraic space
  - Statistics: if we consider features as random variables and features vectors as random vectors



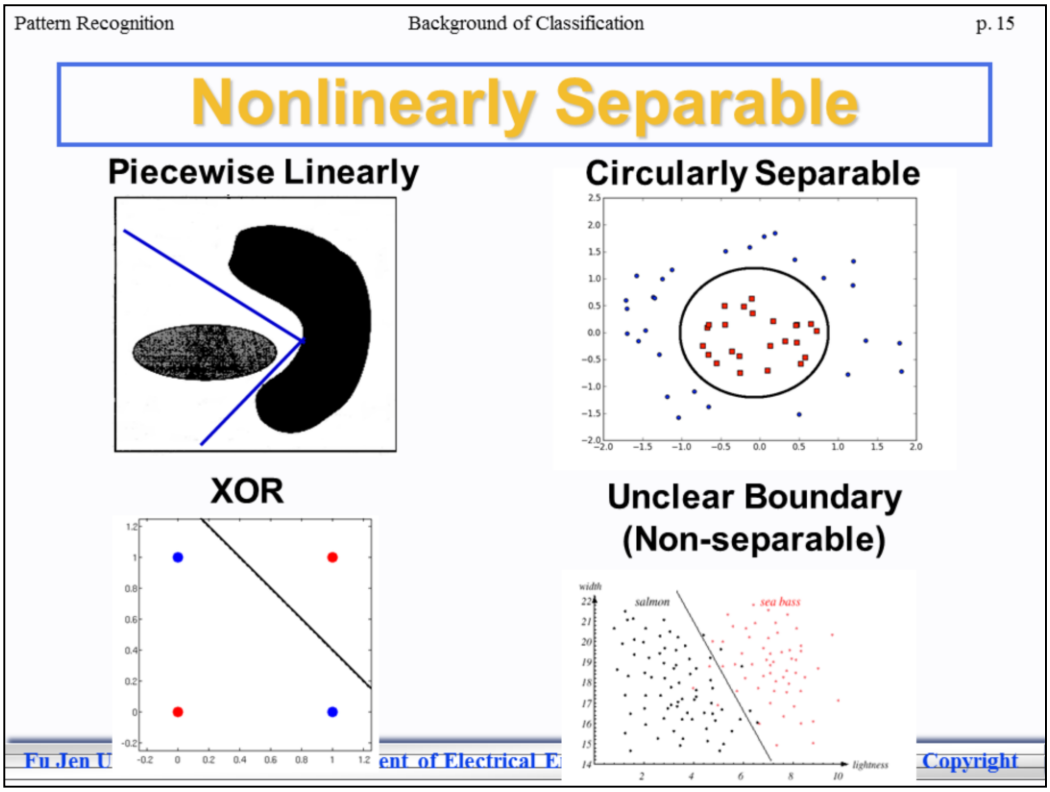

- 2D with 2 classes is a standard case that is only used for concept explanation
    - It is easy to demonstrate basic concepts of pattern recognition
- But standard cases are only toy examples but not real-case examples
    - Sometimes we will show real-case examples
    - Ex.: 2D with more than two classes, 3D with two or more than two classes.




- Three subsections in Section 3
  - 3.1 Linearly separable patterns
  - 3.2 Piecewise-linear separable patterns
  - 3.3 Nonlinearly separable patterns
- A class consists a set of object patterns/object points
- Separable patterns means that there exist lines, curves or hyperplanes that discriminate the classes
- Linearly separable patterns belong to linear classification problem.
  - Linear => straight line (2D), because a straight line is a linear function
  - Linear => plane (3D), because a plane is also a linear function
  - For more than 3D cases, linear => hyperplane.
  - All first-order polynomials are linear
- Nonlinearly separable patterns belong to nonlinear classification problem.
  - Circle is a nonlinear function
  - Ellipse is a nonlinear function
  - Parabola and hyperbolic curves are nonlinear functions
  - All high-order polynomials are nonlinear




- Here we can see three examples of linearly separable patterns
  - The first row has two 2D examples
  - The second row shows a 3D example
- In a 2D space, the line to linearly separate classes are called a linear classifier
  - The classifier can be described by $ax + by + c = 0$, if the $x_1$ is regarded as $x$, and $x_2$ is regarded as $y$.
  - The classifier is also called a decision surface, decision line, or decision boundary.
- In a 3D space, the plane to linearly separate classes are still called a linear classifier
  - A plane can still be described by a linear formula: $ax + by + cz + d = 0$.



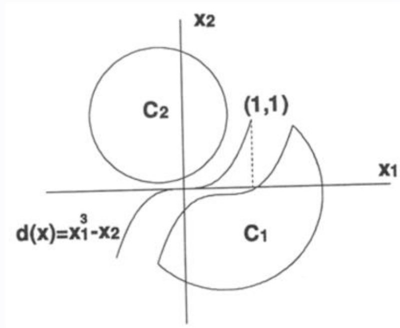

- Patterns in feature space are usually not linearly separable
- Here we give some examples of nonlinearly separable patterns
    - Piecewise linearly separable
    - Circularly separable
    - XOR
    - Unclear boundary
- Piecewise linearly separable
    - Classes can not be separated by only one lines, but can be separated by more lines (pieces of lines).
- Circularly separable
    - Classes have clear boundary, but they can not be separated by lines. Circle or ellipse can be used to separate these two classes.
- XOR
    - Two classes, red and blue, can not be separated by any linear formulas.
    - This is a very famous case for pattern classification. Later we may see more discussions of this XOR case.
- Unclear boundary (for the fish classification example)
    - The previous three examples are nonlinear but with clear boundary between classes.
    - For the fish classification example, there is no clear boundary between two classes. Therefore a line is not able to separate the two classes.
    - This example is very close to many real cases in pattern recognition systems. More advanced pattern recognition algorithms are proposed for this kind of nonlinearity.

# 3.1 Linearly Separable

- If classes are separable with both
  - Linear discriminant functions and
  - Nonlinear discriminant functions,
- It is called linearly separable

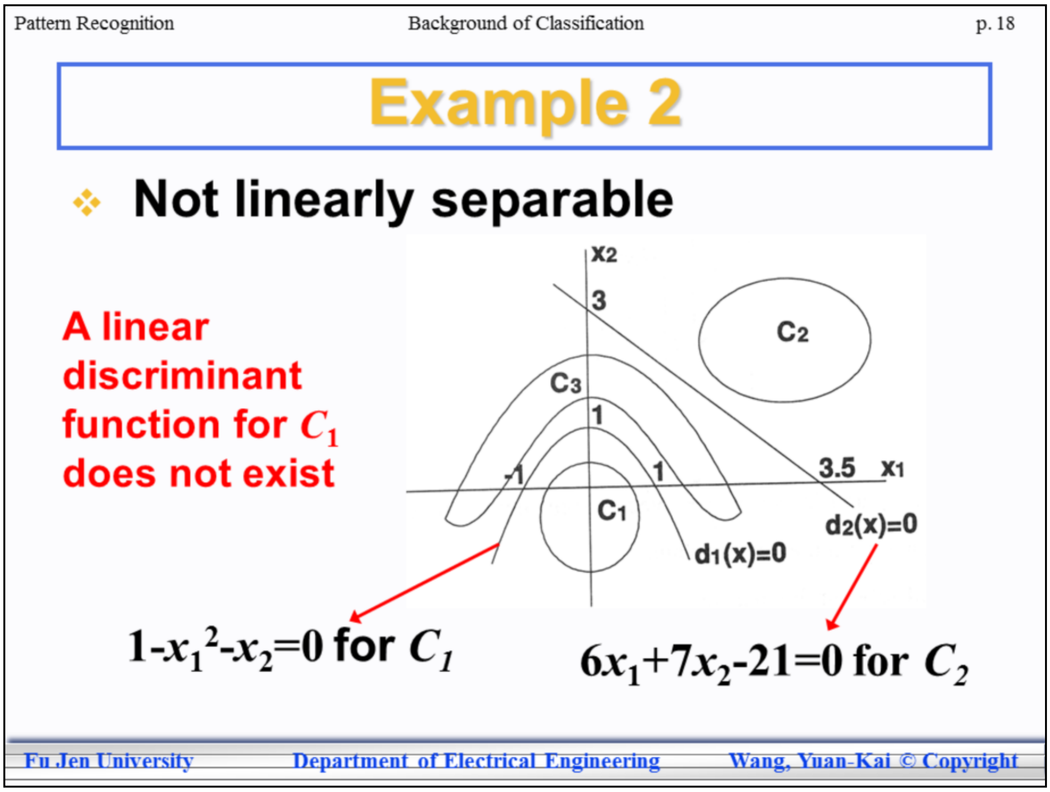
# Example 1

- A linearly separable example
  - Linear discriminant function: straight line $x$-$y$=0
  - Nonlinear discriminant function: parabola $x_1^3 - x_2 = 0$

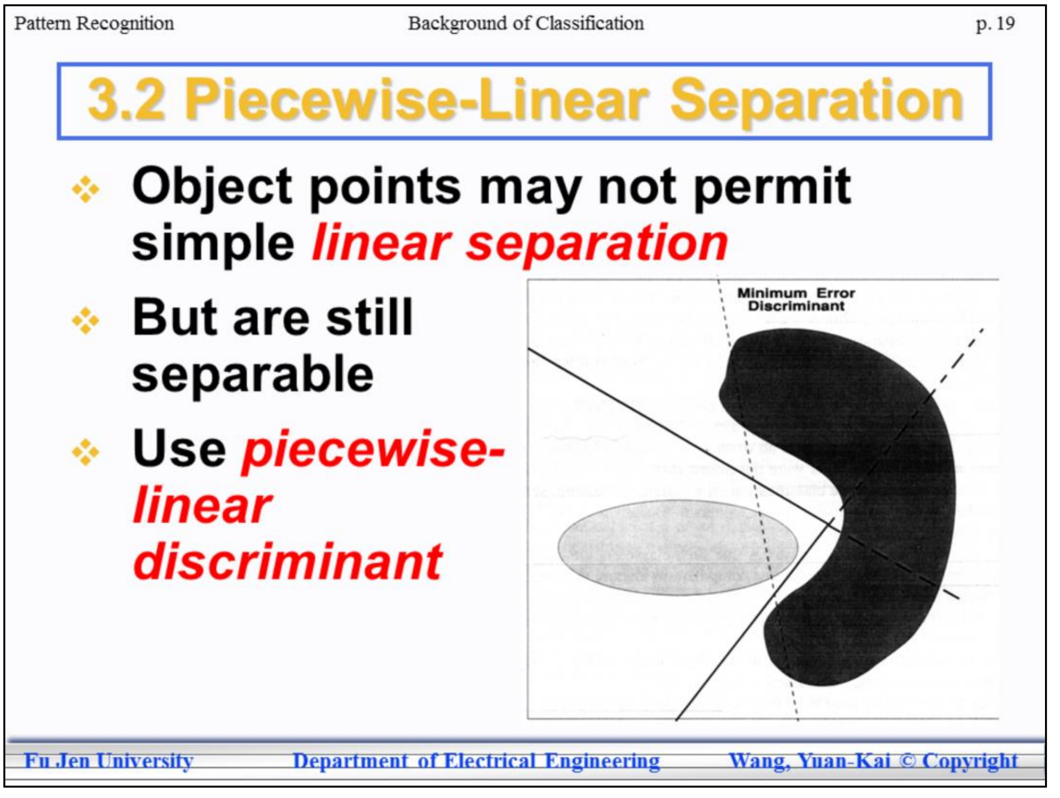





- For this example, both linear discriminant and nonlinear discriminant can separate patterns
  - C1 region represents a lot of pattern points that belong to the class 1
  - C2 region represents a lot of pattern points that belong to the class 2
  - The line *x-y=0* can separate the two classes
  - A parabola can also separate the two classes
- However
  - This example should be called a linear separable example
  - That is, if "at least one" linear discriminant exists for the example, then the example should be called "linear separable"



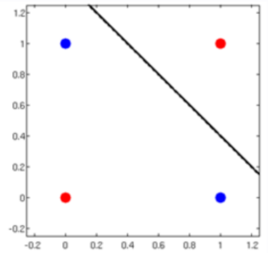

- For this 3-class example, it is called nonlinearly separable, because
    - No linear discriminants exist to separate classes 2 and 3.
    - Only nonlinear discriminants can solve this example.
- Linear discriminants are easier than nonlinear discriminants
    - Always use linear discriminants first to separate patterns
    - If it is not possible to use linear discriminants, then we still can use nonlinear discriminants




- Piecewise-linear discrimination:
    - The study of linear discrimination in the past was very popular in academic circles because it was easy to produce iterative learning algorithms.
    - So, in the early years of PR research a great deal of attention was devoted to *separable* problems.
        - Separable problems are the problems in which discriminants could be found that gave **error-free separation** of points in pattern space.
- However, piecewise-linear discrimination
    - is used only in simulation presented in classroom explanation.
    - is useless in real cases because of the presence of nonconvex and noncompact clusters.

# XOR Problem

- Two classes
  - Each class has **two clusters**
- The Sebestyen problem is very similar to the XOR problem
- XOR problem is not linear separation, but it can be solved by piecewise linear

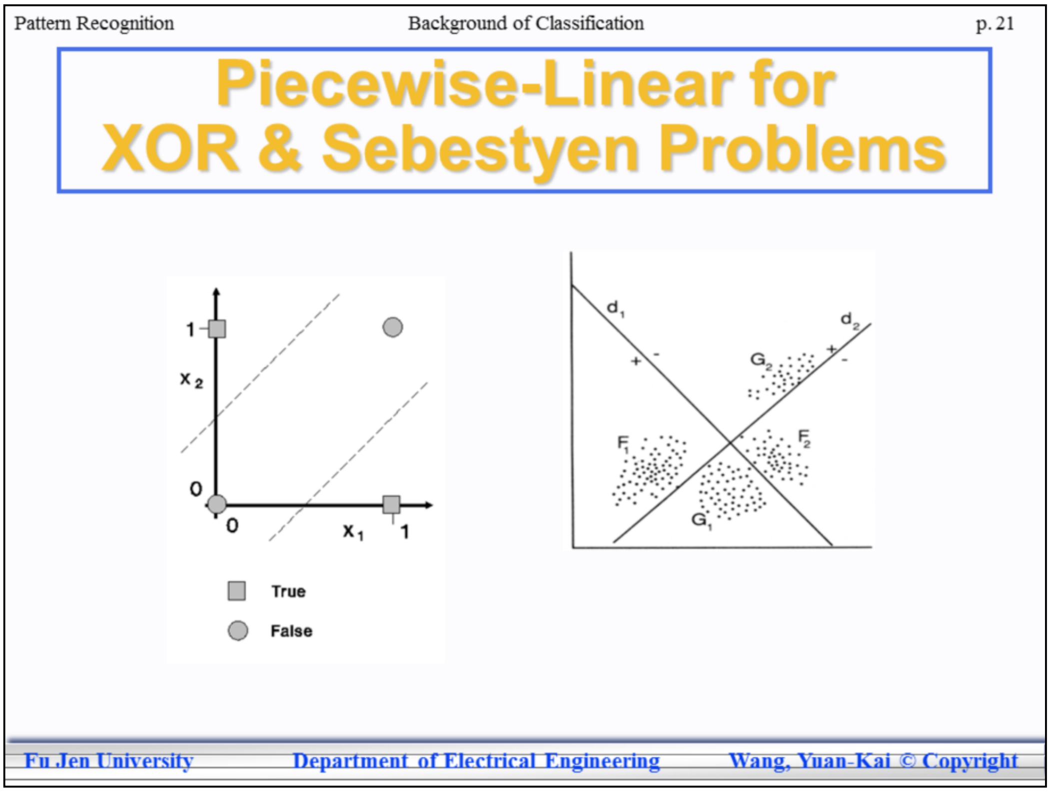


***The Sebestyen Problem***

Two classes: F and G

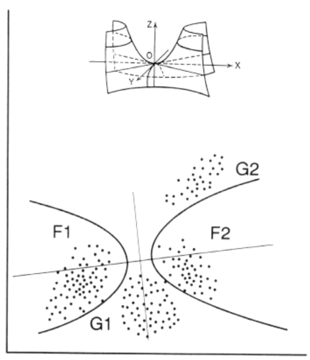


- The Sebestyen problem is proposed by Sebestyen in 1962
  - The problem consists 2 classes, each composed of 2 subclasses.
  - No linear discriminant exists that will separate the classes.
  - Sebestyen believed that the discriminant of lowest degree that will separate them is of 6th degree.
- The XOR problem is proposed by Minsky in 196x.
  - The problem consists 2 classes, each composed of 2 points.
  - The XOR is a very simple case of the sebestyen problem: each cluster has only one point.
  - It is called XOR because it corresponds to the XOR logic operation
    - 0 XOR 0 gets 0
    - 1 XOR 1 gets 0
    - 1 XOR 0 gets 1
    - 0 XOR 1 gets 1
  - If we consider the simple XOR logic operation to be a pattern classification problem
    - It is actually not a simple PR problem.
    - It can not be solved by linear separation, but only by piecewise or nonlinear separation.




- Piecewise linear separation
  - 2 linear discriminants can separate all of the subclasses from each other.
  - Each discriminant will yield one decision, labeled in the diagram "+" or "-".
- However, nonlinear separation is also possible to separate these two problems
  - Ex.: A quadratic discriminant, such as a hyperbolic parabola, is able to separate the XOR and Sebestyen's problems

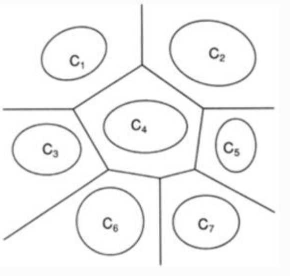
$$\frac{(u+a)^2}{c} - \frac{(v+b)^2}{d} = k$$




Where $u=mx+ny$ and $v=px+qy, a,b,c,d,k,m,n,p,q$ are constants.

# Linearly Separable *N*-class Problem

- For an *N*-class classification problem
- We can reduce it into *N* 2-class problems
  - Reduces the complexity
  - N piecewise linear problems

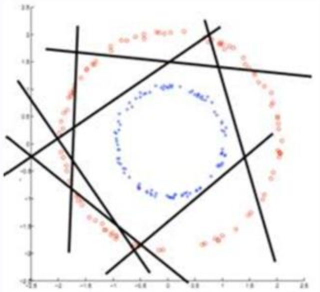


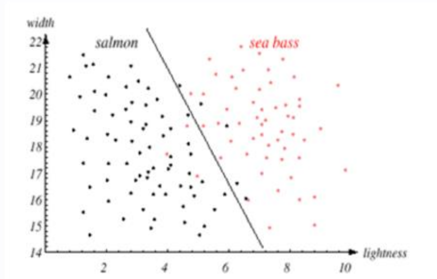



- Previous discussions concern only 2-class problem
- For N-class problems, we need to use "Divide and conqure" to reduce the complexity of an N-class problem
  - (A) Considering *n* different problems: class $C_i$, 1 <= *i* <= *n*.
  - (B) If these classes are linear-dependent, they can be decomposed into *n* 2-classes.
  - (C) If the no. of discriminants in the general *n*-class problem could grow as $n^2$, in the decomposed approach the no. of discriminants will grow as *n*.

# 3.3 Nonlinearly Separable Patterns

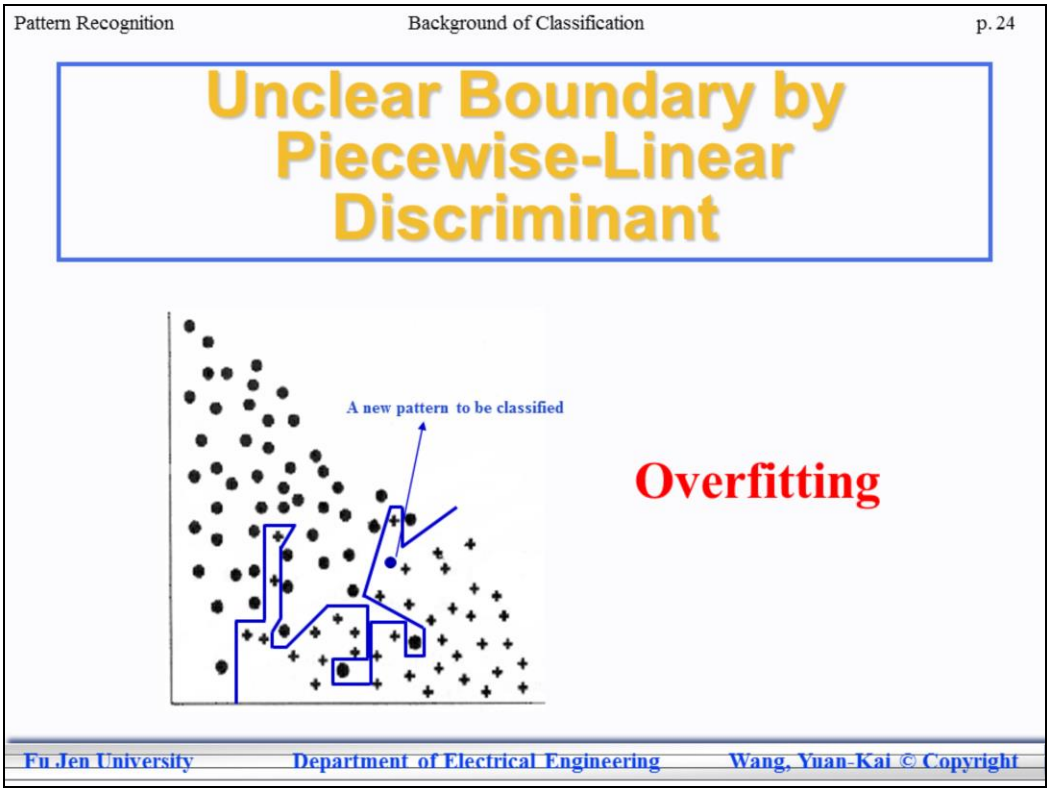
- Some patterns can not be solved by linear and piecewise linear
- Only nonlinear discrimination is possible



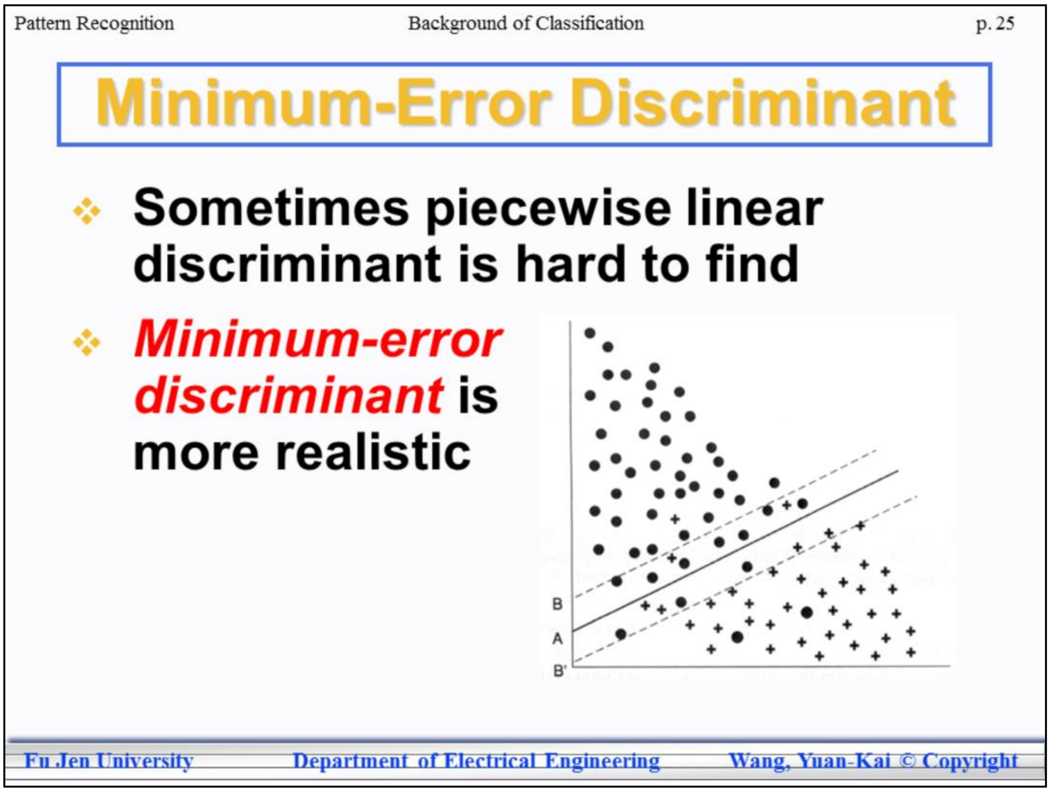





- Left example: Circular patterns can not be separated by linear and piecewise ways.
- Right example: Unclear boundary can not be separated by linear and piecewise ways.



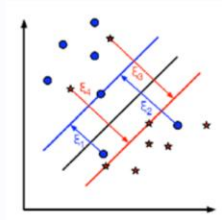

- Sometimes it is possible to separate unclear-boundary patterns with piecewise linear discriminant
- But it is called overfitting and it is not good for pattern classification
    - For a new pattern, the blue circle dot, it is mis-classified
    - That is, although the piecewise linear lines are "perfect" for "learning data", it is still possible tomis-classify unknown patterns. It is not "perfect" actually.
    - For a "perfect" learned classifier that is "actually not perfect", we call it "overfitting".
- Conclusion
    - For unclear boundary problems, it is nonsense to get a perfect classifier
    - All we need to get is to get a "minimum error" classifier
    - Or we can use other complicated methods



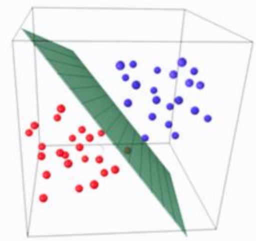

- Linear separation emphasizes on "linear separability"
  - It is assumed that there was no overlap of clusters of different classes.
  - However, real-world problems usually contain overlapped clusters.
- Non-separable patterns means overlapped patterns
  - No **perfect** linear/piecewise-linear discriminant exists.
  - A good way is to choose a linear/piecewise-linear discriminant **with the munimum error**.
- **The minimum-error discriminant**
  - It is usually the case that practical applications will produce **overlapping clusters**.
  - If it is less costly to reject rather than to make an error, we could use the paired discriminants shown in dotted line.
  - Error is usually undetected, while reject is usually processed "manually," that is, by human being.

# Separate Nonlinear Patterns

- Two methods
  - Minimum distance (minimum error)



  - Increase dimensions of features
    - Add more features
    - Transform N dimensions into N+1 without adding features (ex.: SVM classifier)

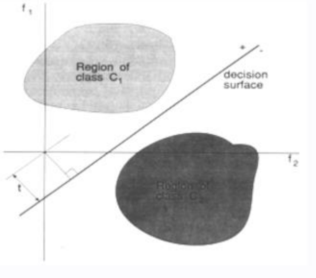


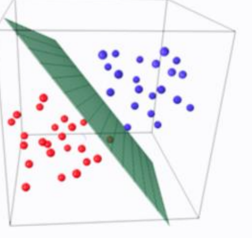


- Why to increase dimensions of features
  - Two-class patterns in 2D may overlap and become a unclear-boundary nonlinear problem
  - Usually by adding more features, ex, add one more feature to become a 3D feature space, those patterns become separable
    - Of course, the one more feature should be more discriminative to classify the two classes.
- How to increase the dimensions of features
  - Real features
    - Extract real features from images
  - Simulated features
    - Use mathematical ways, such as transform, to increase the number of features.
    - A very well-known method: SVM (support vector machine), uses this way to get very good classification results for many PR problems.

# 4. Discriminant and Classifier

- A discriminant is
  - The line(plane, hyperplane) or curve (surface, hypersurface) that separates/discriminate two classes
  - Also called a classifer, decision surface



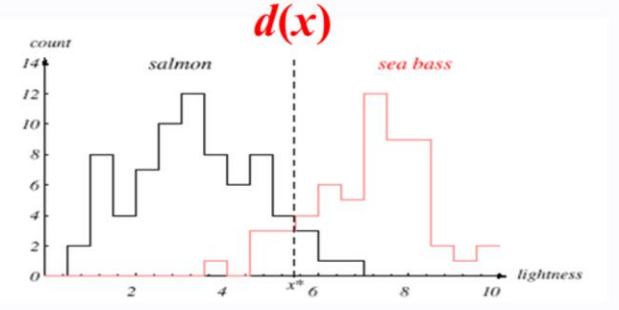





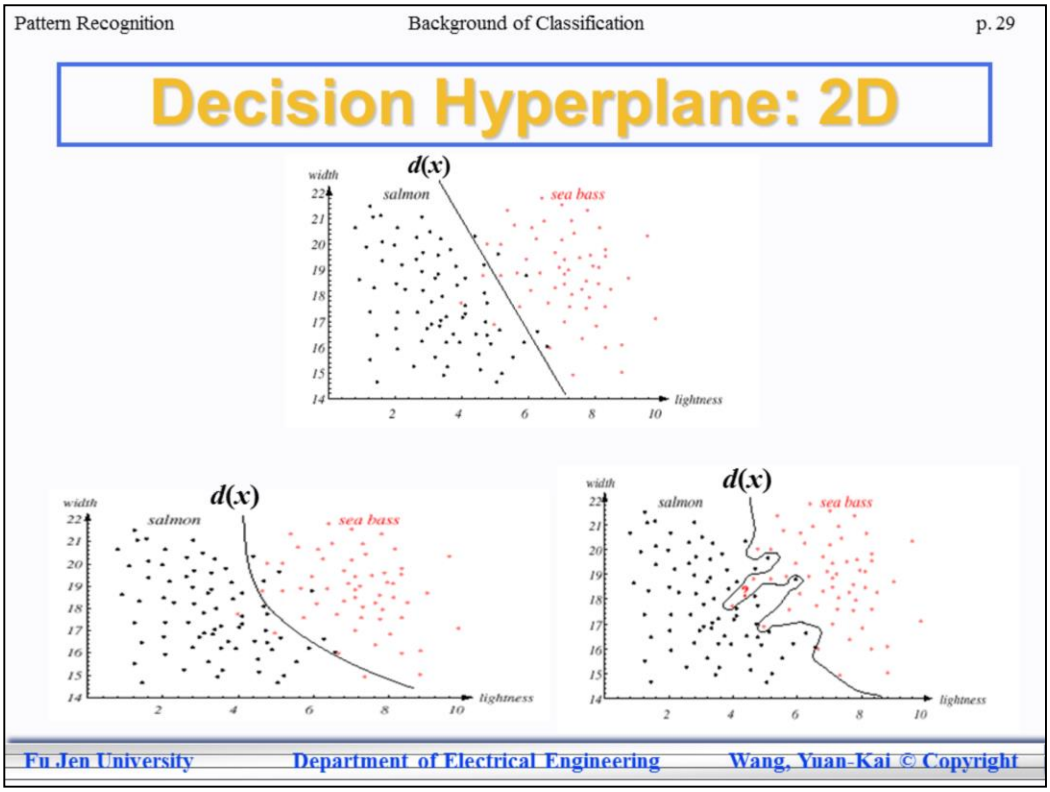
- Section 4 has three sub-sections
  - 4.1 Linear discriminant
  - 4.2 Multi-class discriminant
  - 4.3 Nonlinear discriminant

# Decision Hyperplane: 1D

- **Decision hyperplane is the decision boundary in higher dimensions (*D* > 2)**



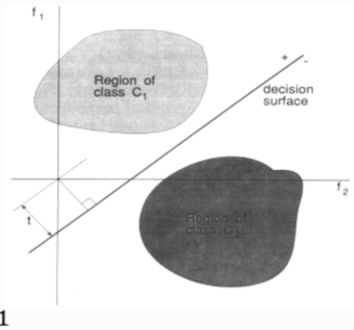

The decision boundary for *D*=1 is also called a **threshold**




- Linear case
  - This decision boundary is also called a "straight line"
  - It is also called linear classifier, such as
    - Minimum distance classifier
    - Perceptron
- Nonlinear case
  - This decision boundary is also called a "curve"
  - It is also called nonlinear classifier
    - Bayes classifier
    - Support vector machines (SVM)
    - Backpropogation, Decision tree, …
- For $D$>2, we call the
  - Linear decision surface as a "decision hyperplane"
  - Nonlinear decision surface as a "decision surface"

# 4.1 Linear Discriminant

- A straight line could be the "separation surface"
- 2D case




Line $w_1 f_1 + w_2 f_2 = w_0$

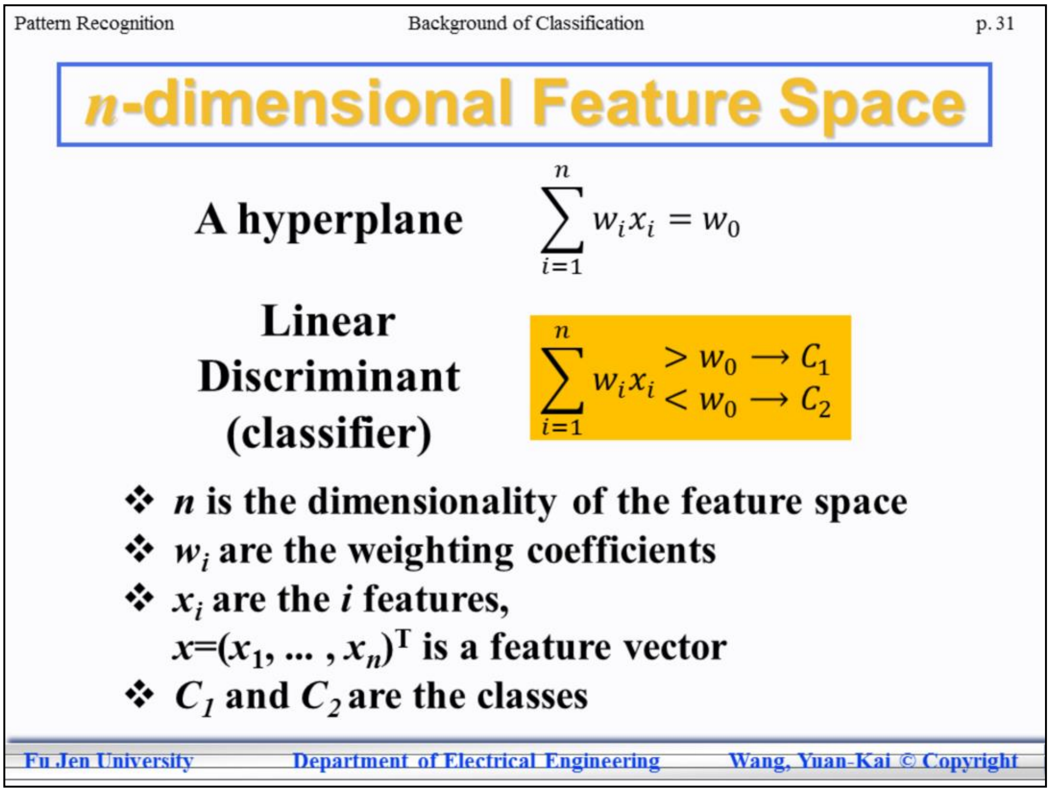
Linear Discriminant (classifier)

$$\sum_{i=1}^{2} w_i f_i > w_0 \Rightarrow (f_1, f_2) \in C_1$$

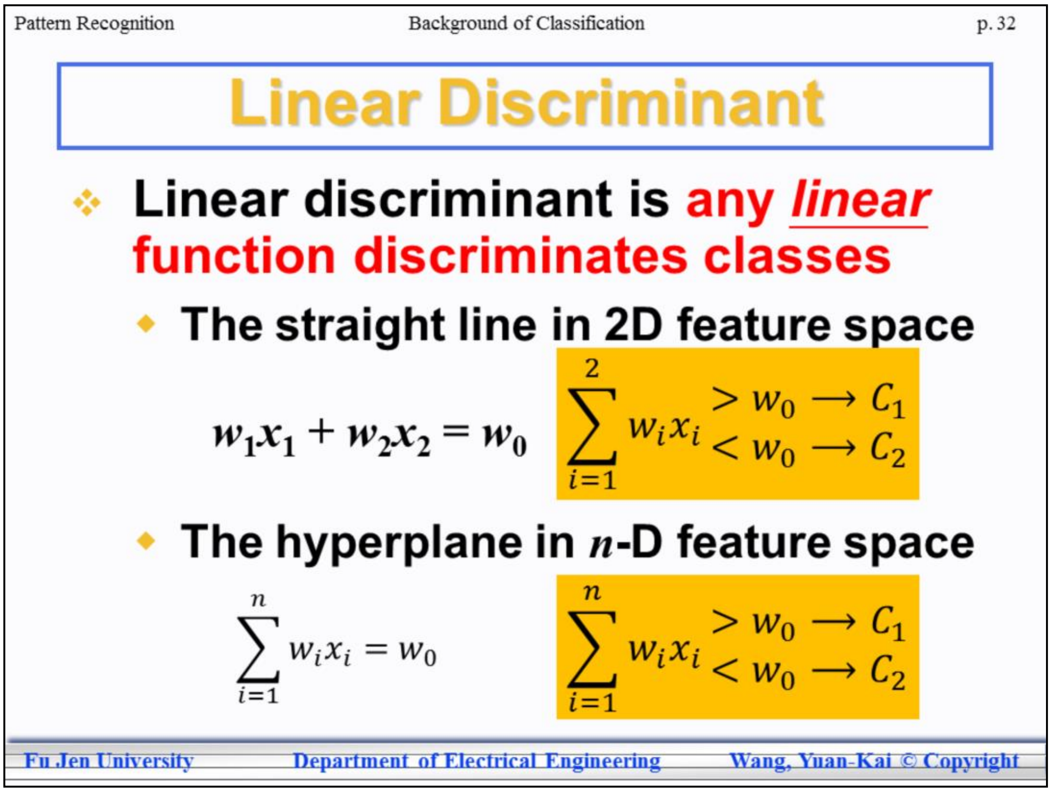
$$\sum_{i=1}^{2} w_i f_i < w_0 \Rightarrow (f_1, f_2) \in C_2$$



- Suppose
  - Only two classes C1 and C2
  - Only two features: f1 and f2
  - A pattern (image object) is represented as the coordinates (f1, f2)
- The straight line to discriminate C1 and C2 is called a linear discriminant function.
  - When a hyperplane/hypersurface separates 2 clusters, the function that defines it is called a *discriminant*.
  - The functional form of a discriminant is an equation with
    - The coefficients and variables of the space on the left side
    - Zero on the right side.
  - Discriminant is the locus of all points that satisfy the equation
- The best well-known linear discriminant is called "fisher" classifier.



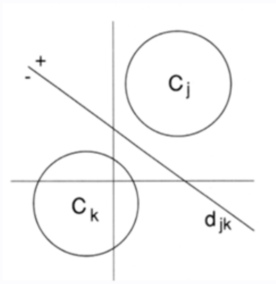

- Here we extend the two-feature case ($n$=2) to more-feature case: $n$>2.
- The feature space is then extended into $n$-D: ($x_1$, $x_2$, ..., $x_n$)
  - We replace the symbol $f_1$, $f_2$ with $x_1$, $x_2$ for generalization.
- A hyperplane is a linear equation in n-dimensional space for $n$>2.
  - Remember that a linear equation in 2D is called a straight line.
- A hyperplane is an equation, and a discriminant is an inequality.




- The slide gives a quick comparison between 2D and $n$-D cases. All formula have appeared in previous two slides.

# Matrix Form of Hyperplane

- **The hyperplane can be rewritten by the matrix form**




$$\sum_{i=1}^{n} w_i x_i + w_0 = 0$$

$$\Rightarrow w_n\, x_n + w_{n-1} x_{n-1} + \cdots + w_1 x_1 + w_0 = 0$$

$$\Rightarrow W^T\, x = 0\,, W = \begin{pmatrix} w_n \\ \vdots \\ w_1 \\ w_0 \end{pmatrix} \; x = \begin{pmatrix} x_n \\ \vdots \\ x_1 \\ 1 \end{pmatrix}$$

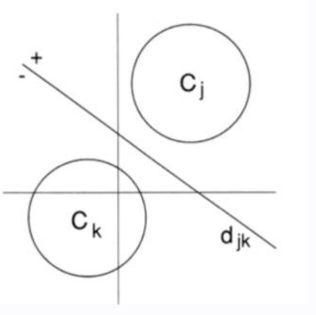


- After the understanding of discriminant with the formula of a basic form, we want to rewrite the formula of discriminant into a matrix form.

# Matrix Form of Linear Discriminant

- The linear discriminant can be rewritten by the matrix form

$$\forall \boldsymbol{x} \in \text{feature vector}$$
$$\begin{cases} \boldsymbol{x} \in \boldsymbol{C_j} \ \ if\ \boldsymbol{W^T x} > 0 \\ \boldsymbol{x} \in \boldsymbol{C_k} \ \ if\ \boldsymbol{W^T x} < 0 \end{cases}$$






- Here we successfully apply the Linear Algebra, a good mathematic tool, to present the linear discriminant.
- That means linear algebra is very helpful for us to know more of linear discriminants, if we proceed to learn more of linear discriminant.
  - However, in this unit, we do not go deep into more of linear discriminant.
  - In this unit I just give you a basic understanding of linear discriminant.

# 4.2 Multi-class Discriminant

- Extension to $m$ classes $\{ C_1, C_2, \ldots, C_m \}$, or $\{C_i\}^m_{i=1}$
  - Let $C_1, C_2, \ldots, C_m$ be the $m$ classes
- There are two kinds of separation
  - Absolute separation
  - Pairwise separation

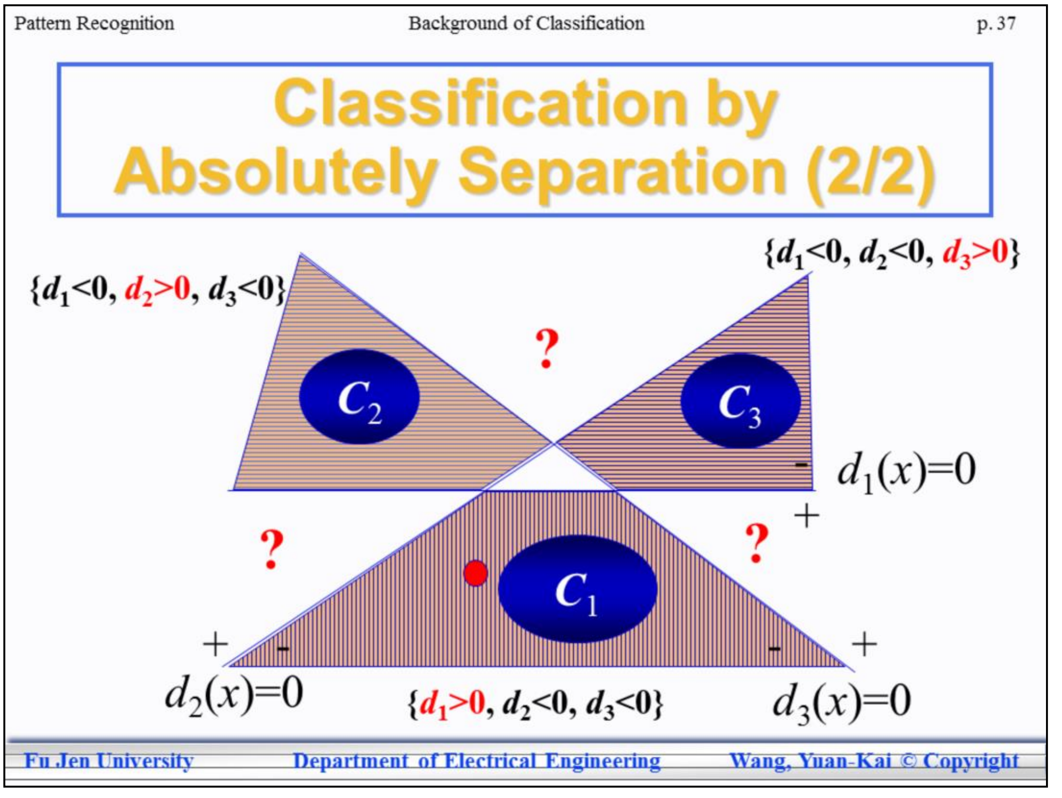
# Absolute Separation (1/2)

- If **each** pattern classes $C_i$ has a linear discriminant function $d_i(x)$

$$d_i(x) = W_i^T x = \begin{cases} > 0, x \in C_i \\ < 0, \text{otherwise} \end{cases}$$

It is called absolute separation

- We have $m$ discriminant functions $d_1(x), d_2(x), \ldots, d_m(x)$
- There are $m$ discriminant regions $D_i$
  $D_i = \{ x \mid d_i(x)>0; d_j(x)<0, j \neq i \}, 1 \leq i \leq m$
- If $x$ locates in $D_i$, ie. $D_i>0$, then $x \in C_i$




- In the example for three classes, $m$=3.
- We have *3* discriminant functions $d_1(x)$, $d_2(x)$, $d_3(x)$
- There are *3* discriminant regions
  $D_i = \{ x \mid d_i(x)>0; d_j(x)<0, j \neq i \}, 1 \leq i \leq 3$
- If $x$ locates in $D_i$, ie. $D_i$>0, then $x \in C_i$

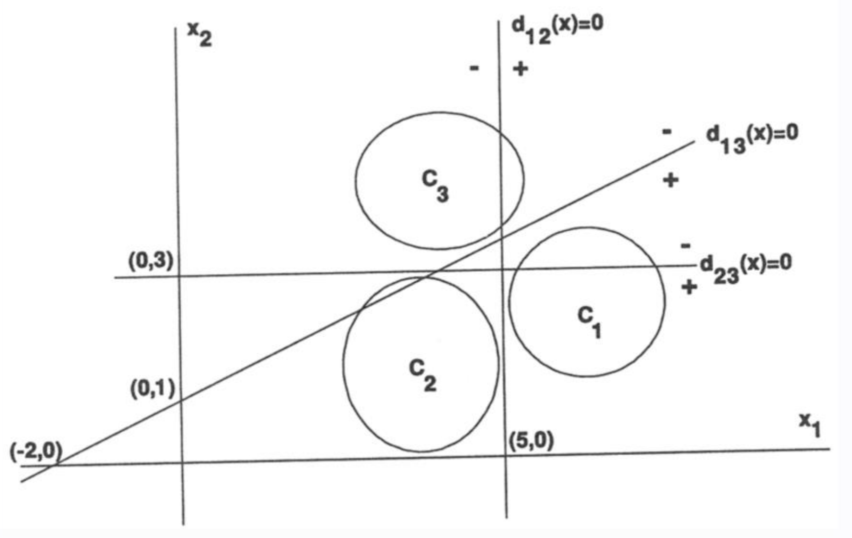
# Pairwise Separation

- If there is no absolutely separation
- But **each pair of classes $C_i$ and $C_j$ are associated** with a linear discriminant function $d_{ij}$, such that
  - $d_{ij}(x) > 0$ for all $x \in C_i$
    $d_{ij}(x) < 0$ for all $x \in C_j$
- $d_{ij}(x) = - d_{ji}(x)$

# Classification by Pairwise Separation

- Given ***pairwise separable classes*** $\{C_i\}^m_{i=1}$, how do we classify an input $x$?
  - $x \in C_i$, if and only if $d_{ij}(x) > 0$ for all $j \neq i$

# A Pairwise Separable Example



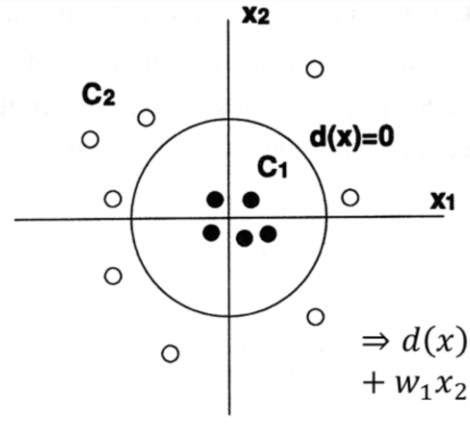
# 4.3 Nonlinear Discriminant

- We take two approaches as examples
  - SVM (support vector machine)
  - NN (neural network)



- There are a lot of nonlinear discriminants
  - Bayesian classifier
  - SVM
  - NN: backpropagation, deep neural network, ...
  - Adaboost
- SVM and NN are the two popular nonlinear classifiers in recent years.

# Nonlinear Discriminant Function




$$d(x) = 1 - x_1^2 - x_2^2 = 0$$

$$x \in C_1 \text{ if } d(x) > 0$$
$$x \in C_2 \text{ if } d(x) < 0$$

$$\Rightarrow d(x) = w_5 x_1^2 + w_4 x_2^2 + w_3 x_1 x_2 + w_2 x_1 + w_1 x_2 + w_0 = 0$$

$$\Rightarrow d(x) = W^T x = 0$$

$\forall x \in$ **feature vector**

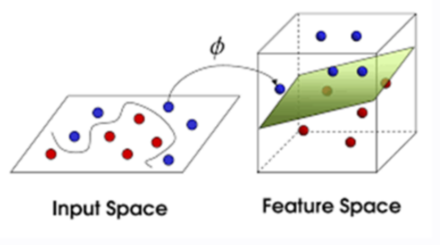
$$\begin{cases} x \in C_1 \text{ if } W^T x > 0 \\ x \in C_2 \text{ if } W^T x < 0 \end{cases}$$

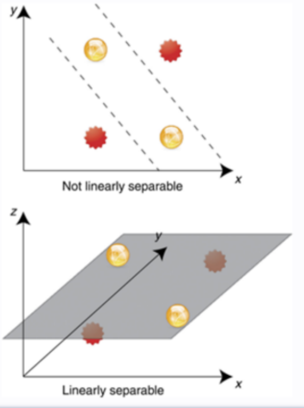


- This is a special case of nonlinear equation *d*(*x*), just for circular separable patterns.
- The nonlinear equation is a circle. It corresponds to a discriminant.
- In the right bottom, I write a new derivation of the *d*(*x*) and the discriminant into a normal form
    - You should be able to write *w*5, *w*4, *w*3, *w*2, *w*1, and *w*0 by yourself.
    - Could you write the *W* vector and the *x* vector by yourself?

# SVM

- Transform **low**-dimensional **nonlinearly** separable patterns into **high**-dimensional **linearly** separable patterns



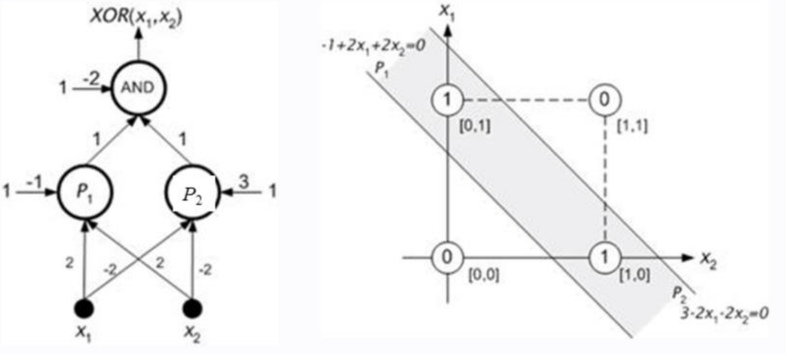






- Nonlinearly separable patterns in low dimensions can be linearly separable in high dimensions,
- SVM fully applies this concept to classify very difficult problems:
  - First step: transform all patterns into higher dimensional feature space.
  - Second step: apply linear classifier to recognize patterns in the higher dimension.

# NN

- Cascade of linear classifiers into a nonlinear classifier






- Please see the online book for details
  - Celebi Tutorial: Neural Networks and Pattern Recognition Using MATLAB (https://www.byclb.com/TR/Tutorials/neural_networks/)
    - Chapter 8 Classical Models of Neural Network

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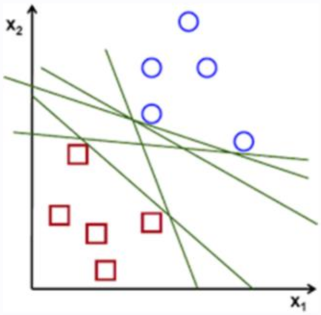
# 5. Find the Best Classifier

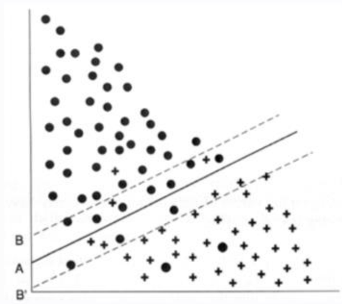
- **Suppose we want to find linear classifiers for both linear and nonlinear patterns**

**Linear patterns**




Which line is the best?

**Nonlinear patterns**




Which line is the best?

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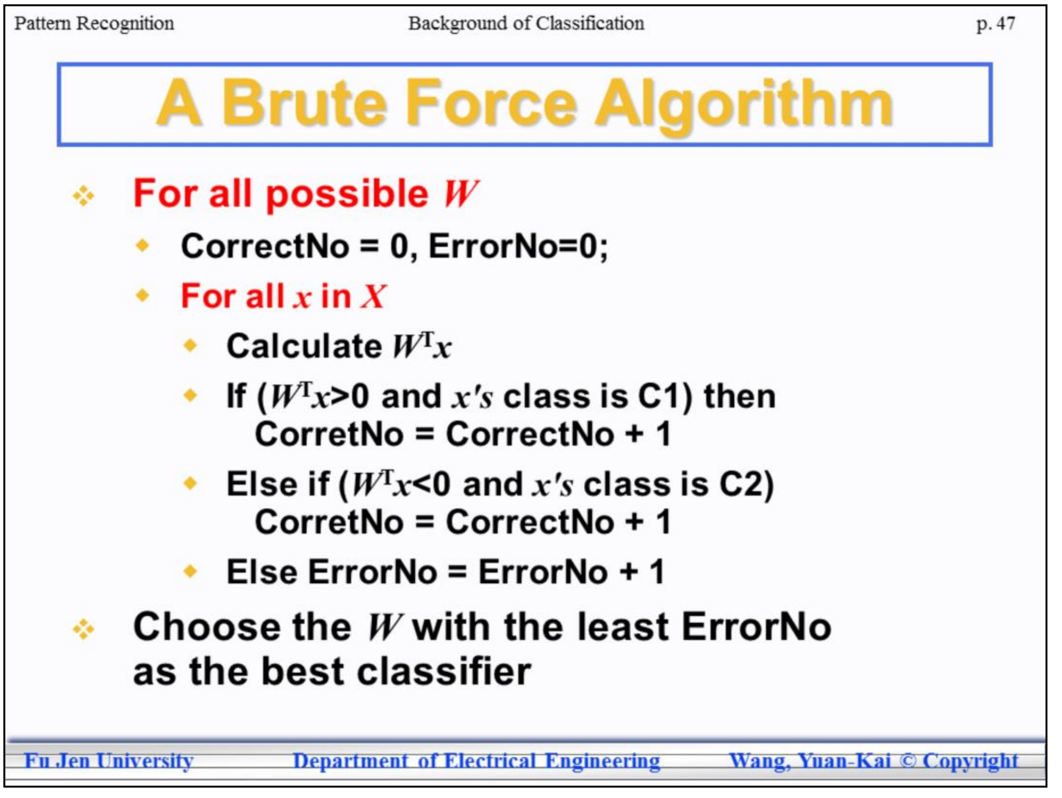
- Section 5 introduces “machine learning” for the finding of the best classifier.

# The Machine Learning Problem

- For a 2-class problem: classes C1 and C2
- To find a best linear classifier $W=(w_0, w_1, \ldots, w_n)^T$
- We need a set of learning data $X=\{(x^{(1)},C1), (x^{(2)},C1), \ldots, (x^{(K)}, C2)\}$
- We then apply a machine learning algorithm to find the solution



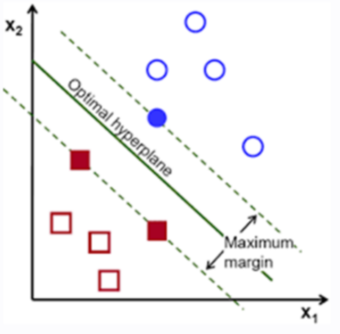
- To find the classifier of a classification problem is also called a machine learning problem.
- A machine learning algorithm can either
  - Find a possible classifier, or
  - Find the best classifier




- A brute force method is a bad but simple approach to find the best classifier.
- It works like "try and error". It works straightly.
- But it take too many times to find the solution: the best classifier.
- So there are other better machine learning algorithms:
    - SVM, NN, Bayesian classifier, ...
- How do think about this brute force algorithm?
    - Is it efficient or is it time consuming? Is there any other algorithms that works better?
    - Could it find the best classifier? Or it just finds possible classifiers.

# Linear Cases

- The line with maximum margin is the best

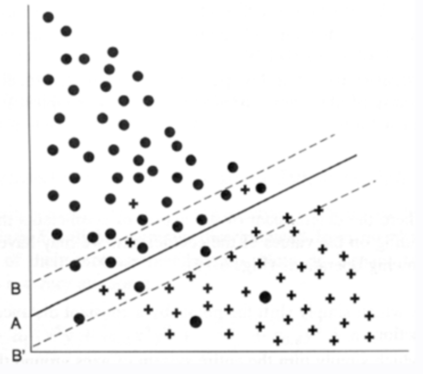





- Maximum margin is a good criteria to define "the best" classifier in linear cases.

# Nonlinear Cases

- The line/curve with minimum error is the best






- Minimum error is one of good criterion to define a "best" classifier for nonlinear cases.

# Conclusion

- Patterns in feature space
  - Linearly separable vs. nonlinearly separable
  - Real-case patterns are nonlinearly separable
- Discriminant and classifier
  - Linear discriminant vs. nonlinear discriminant
- Machine learning is helpful to
  - Find possible classifiers
  - Find the best classifier